Analysis SEP 2026: Topology

1 Warm-up

Problem 1.07.1 Let X be a metric space with metric d.

(i) Define $\rho : X \times X \to \mathbb{R}$ by
$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$
Prove that ρ is a metric on X.

(ii) Show that a subset U of X is open with respect to the metric d if and only if it is open with respect to the metric ρ.

2 Exercises

Problem 8.19.3 Prove that if K is a subset of $\mathbb{R}^n$ such that every continuous real-valued function on K is bounded, then K is compact.

Problem 1.23.1 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume the Jacobian matrix $(\partial f_j / \partial x_j)$ has rank n everywhere. Suppose f is proper; that is, $f^{-1}(K)$ is compact whenever K is compact. Prove $f(\mathbb{R}^n) = \mathbb{R}^n$.

(**Hint from ya gurl:** $\mathbb{R}^n$ is a connected space, so the only clopen (= closed and open) sets in $\mathbb{R}^n$ are $\varnothing$ and $\mathbb{R}^n$.)
Note: This result holds more generally for smooth manifolds M, N where N is connected.

Problem 1.17.3 Let $\ell^1(\mathbb{N})$ be the space of all summable sequences, i.e.
$$\ell^1(\mathbb{N}) = \{x : \|x\|_{\ell^1} < \infty\}, \quad \text{where} \|x\|_{\ell^1} \overset{\text{def}}{=} \sum_{n=1}^{\infty} |x_n|.$$
Let a_n be a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}$, and consider the subset $K \subset \ell^1(\mathbb{N})$ defined by
$$K = \{x \in \ell^1(\mathbb{N}) : 0 \leq x_n \leq a_n \text{ for all } n \in \mathbb{N}\}.$$
Show that K is compact if and only if the sequence $\{a_n\}_{n=1}^{\infty}$ itself belongs to $\ell^1(\mathbb{N})$.

Problem 8.22.6 Suppose $E \subset \mathbb{R}^n$ is a given set and O_n is the open set $O_n = \{x : d(x, E) < 1/n\}$.

a) Show that if E is compact, then $|E| = \lim_{n\to\infty} |O_n|$.

b) Is the statement false for E closed and unbounded?

c) Is the statement false for E open and bounded?

Problem 8.11.3 Let E denote the vector space of bounded sequences of real numbers $x = (x_1, x_2, \ldots)$. Define norms

$$\|x\|_1 = \sup |x_n|, \quad \|x\|_2 = \sup \frac{|x_n|}{n}.$$

Let B be the set of x in E with $\|x\|_1 \leq 1$.

(i) Prove or disprove that B is compact in $(E, \|\cdot\|_1)$.

(ii) Prove or disprove that B is compact in $(E, \|\cdot\|_2)$.

3 Homework

Solve the following problems.

- 1.23.3
- 8.22.2
- 8.22.3
- 1.14.3
- 1.02.3
- 1.22.1